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EMBEDDINGS IN ARTINIAN RINGS AND SYLVESTER RANK FUNCTIONS

BY

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ABSTRACT

Using Sylvester rank functions it is shown that a right Noetherian algebra, modulo the torsion ideal determined by elements regular modulo its nil radical, is embeddable in a simple Artinian ring. This is used to show that a right Noetherian algebra which is right Krull homogeneous embeds in a simple Artinian ring, and that a right Noetherian affine algebra satisfying a polynomial identity embeds in a simple Artinian ring.

1. Introduction

Ring theorists have long been interested in determining when a Noetherian ring can be embedded in an Artinian ring. The best known results of this genre are Goldie's Theorems which characterize those Noetherian rings which are orders in simple or semisimple Artinian rings. A theorem of the second author characterizes those Noetherian rings which are orders in Artinian rings. In this paper we will consider Noetherian k-algebras, where k is a field, which for our purposes can always be assumed to be $\mathbf{Q}$ or $\mathbf{Z}_p$, and relax the requirement of a "tailored fit" of R as an order in an Artinian ring, and simply ask whether R can be embedded in an Artinian k-algebra S. Without loss of generality we may require that S be a simple Artinian k-algebra since Schofield [9] has shown that every Artinian k-algebra can be embedded in a simple Artinian k-algebra.

Small [10] has shown that every ring which is finitely generated as a module over a Noetherian subring of its center can be embedded in an Artinian ring. Blair [1] generalized this to show that every right Noetherian ring which is integral over its center is embeddable in a right Artinian ring, and Gordon [4]

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then showed that every two-sided fully bounded Noetherian ring is a subring of an Artinian ring. These embeddings are obtained by showing that the given Noetherian ring can be written as a subdirect sum of a finite number of Noetherian rings which are, in fact, orders in Artinian rings. In our study of Noetherian k-algebras we will replace the use of orders in Artinian rings by Schofield's notion of a Sylvester rank function.

Not all Noetherian rings can be embedded in Artinian rings. On the negative side of the ledger we have Small's example [11] of a right Noetherian ring which cannot be embedded in a ring which is Artinian on either side. Recently Dean and Stafford [3] have shown that a particular factor ring of the universal enveloping algebra of $\mathrm{sl}(2, \mathbf{C})$ is a two sided Noetherian algebra which cannot be embedded in an Artinian ring.

2. Definitions and notation

Schofield [9] defines a Sylvester module rank function λ for the ring R to be a function which assigns to each finitely presented right R-module M a nonnegative number in $(1/n)\mathbf{Z}$, for some fixed positive integer n, such that

(i) $\lambda(R)=1$,

(ii) $\lambda(M \oplus N)=\lambda(M)+\lambda(N)$,

(iii) if $L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence, then $\lambda(N) \leqq \lambda(M) \leqq \lambda(L)+\lambda(N)$,

for all finitely presented R-modules L, M and N. If (ii) and (iii) are replaced by the stronger condition that $\lambda(M)=\lambda(L)+\lambda(N)$ whenever $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of finitely generated R-modules, then the Sylvester module rank function λ is said to be exact. The main tool in our work is the following important result of Schofield. Throughout the paper let k be a prime field, that is $k=\mathbf{Q}$ or $k=\mathbf{Z}_p$ for some prime p.

THEOREM. *A Sylvester rank function λ on a k-algebra R taking values in $(1/n)\mathbf{Z}$ arises from a homomorphism $f\colon R \rightarrow M_n(D)$, where D is a skew field. The kernel of this map consists of those elements of $a \in R$ such that $\lambda(R/aR)=1$. Furthermore, $M_n(D)$ is flat when considered as left module over R if and only if λ is exact.*

We assume that the reader is familiar with Goldie's notion of reduced rank. The discussion in either Chatters and Hajarnavis [2] or McConnell and Robson [8] is more than ample for our purposes. A ring R is said to be right Krull homogeneous if R and all of its non-zero right ideals have the same Krull dimension. We refer the reader to Gordon and Robson [5] for the relevant facts regarding Krull dimension. We denote the Krull dimension of the right R-module M by $|M_R|$, or more simply $|M|$. We will also have occasion to use the Gelfand–Kirillov dimension of a k-algebra R for which we refer the reader to Krause and Lenagan [6] or McConnell and Robson [8]. We denote the Gelfand–Kirillov dimension of the right R-module M by $GK(M_R)$, or $GK(M)$.

We fix some notation. Let R be a ring (always with unity) and N its nil radical. All ring homomorphisms, in particular all embeddings, will be assumed to carry unity to unity. $C(N)$ denotes the set of all $c \in R$ such that $c + N$ is a regular element of R/N, and $'C(0)$ denotes the set of all left regular elements of R. For a set $X \subseteq R$, $r(X)$ denotes the right annihilator of X and $l(X)$ denotes the left annihilator of X.

3. The results

THEOREM 1. *Let R be a right Noetherian k-algebra, and set $K = \{a \in R \mid$ for each $r \in R$ there exists $c \in C(N)$ such that $arc = 0\}$. Then K is an ideal of R and R/K is a subring of a simple Artinian ring.*

PROOF. Let ρ denote the reduced rank function for finitely generated modules over the right Noetherian ring R. For each finitely generated right R-module M set $\lambda(M) = \rho(M)/\rho(R)$. Since $\rho(M) = \rho(M') + \rho(M'')$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$ of finitely generated right R-modules, λ is an exact Sylvester module rank function taking values in $(1/\rho(R))\mathbf{Z}$. By Schofield's Theorem there exists a homomorphism from R to a simple Artinian ring and the kernel K of this homomorphism is the set $\{a \in R \mid \lambda(R/aR) = 1\}$. For each $a \in R$, $\lambda(R/aR) = 1$ if and only if $\rho(R/aR) = \rho(R)$. Hence $\lambda(R/aR) = 1$ if and only if $\rho(aR) = 0$. Finally $\rho(aR) = 0$ if and only if for each $r \in R$ there exist $c \in C(N)$ such that $arc = 0$ (see Theorem 2.2 of Chatters and Hajarnavis [2]). Hence $K = \{a \in R \mid$ for each $r \in R$ there exists $c \in C(N)$ such that $arc = 0\}$.

The ideal K is the torsion ideal of R determined by the elements regular modulo the nil radical. If $r(K)$ contains an element $c \in C(N)$, then $K = l(r(K))$ will be an annihilator ideal contained in the nil radical, and, thus, frequently smaller than the nil radical. We will refer to the ideal K of Theorem 1 as the Sylvester kernel induced by the reduced rank function of R. The following Corollary is reminiscent of the regularity condition of Small [10]. Here we only require "one hall" of the regularity condition.

COROLLARY 2. *Let R be a right Noetherian k-algebra such that $C(N) \subseteq {}'C(0)$. Then R is embeddable in a simple Artinian ring.*

EXAMPLE. The ring

$$\begin{bmatrix} k[x] & k[x]/(x) \\ 0 & k[x]/(x) \end{bmatrix}$$

is a right and left Noetherian k-algebra which is embeddable in a simple Artinian ring by the criterion of Corollary 2, but which does not have an Artinian ring of quotients.

It is not known whether a right and left Noetherian ring which is right Krull homogeneous must have an Artinian ring of quotients. The next theorem answers the embeddability question in the case of right Noetherian algebras which are right Krull homogeneous.

THEOREM 3. *Let R be a right Noetherian k-algebra which is right Krull homogeneous. Then R embeds in a simple Artinian ring S and ${}_RS$ is a flat left R-module.*

PROOF. Let K be the Sylvester kernel induced by the reduced rank function of R. If $K = 0$, then we are done. Assume $K \neq 0$. By the proof of theorem 9 of Lenagan [7] R satisfies the descending chain condition on right annihilators. Thus $r(K) = r(x_1, x_2, \dots, x_t)$ for some $x_i \in K$. Since $x_1 \in K$ there exists $c_1 \in C(N)$ such that $x_1c_1 = 0$. Suppose that $c_1, c_2, \dots, c_{i-1} \in C(N)$ have been chosen so that $x_1c_1 = 0$, $x_2c_1c_2 = 0, \dots, x_{i-1}c_1c_2 \cdots c_{i-1} = 0$. Pick $c_i \in C(N)$ so that $x_i(c_1c_2 \cdots c_{i-1})c_i = 0$. Let $c = c_1c_2 \cdots c_t \in C(N)$. Then $x_jc = 0$ for $j = 1, \dots, t$ and so $c \in r(K)$.

Among all non-zero right ideals contained in K pick U so that $r(U)$ is maximal. It is easy to check that $P = r(U)$ is a prime ideal of R, and so $P \supseteq N$, the nil radical of R. Note that $c \in r(K) \subseteq r(U) = P$ and so $\bar{P}$, the image of P in $\bar{R} = R/N$, contains a regular element $\bar{c} = c + N$. Thus $|R| = |\bar{R}| > |\bar{R}/\bar{P}| = |R/P|$.

Since U is a finitely generated right R/P-module $|U| \leqq |R/P| < |R|$, contradicting the Krull homogeneity of R. Hence $K = 0$ and the theorem is proved.

In the proof of the previous theorem results on Krull dimension enabled us to show that the Sylvester kernel was zero. In the next theorem we will rely on an interplay between Krull dimension and Gelfand–Kirillov dimension. We recall that R is said to be an affine F-algebra if R is finitely generated as an F-algebra, where F is a field.

THEOREM 4. *Let r be a right Noetherian affine F-algebra which satisfies a polynomial identity. Then R is embeddable in a simple Artinian ring.*

PROOF. Since R is a right fully bounded right Noetherian ring, R admits a primary decomposition by Corollary 2.4 of Gordon [4]. Thus it suffices to show that we can embed R in an Artinian ring in the case where R is P-primary for the prime ideal P.

Let K be the Sylvester kernel induced by the reduced rank of R. We will show that $K=0$. If $K\neq 0$, we choose a nonzero right ideal $U\subseteq K$ such that $r(U)$ is maximal. Then $r(U)$ is prime and since $r(U)=r(U')$ for all nonzero $U'\subseteq U$ we have that $r(U)=P$. By Proposition 1.2 of Gordon [4] $l(P)$ is an essential right ideal of R. Since $l(l(P))$ is contained in the right singular ideal of the right Noetherian ring R, $l(l(P))$ is a nilpotent ideal of R.

Since R is a right fully bounded right Noetherian ring, $r(K)=r(x_1,x_2,\dots,x_t)$ for some $x_i\in K$. Then as in the proof of Theorem 3 there exists $c\in C(N)$ such that $c\in P$, and so $|R/P|<|R|$. Since the ideal $l(l(P))$ is nilpotent, $|R/l(l(P))|=|R|$.

By a theorem of Berele (see Krause and Lenagan, corollary 10.7, [6]), $GK(R)<\infty$, since R is an affine algebra satisfying a polynomial identity. By corollary 3.16 of Krause and Lenagan [6], the Gelfand–Kirillov dimension of an F-algebra is greater than or equal to the length of any chain of prime ideals of the algebra. In any right fully bounded right Noetherian ring the Krull dimension coincides with the maximal length of chains of prime ideals provided such a maximum exists. Thus $|R/l(l(P))|\leqq GK(R/l(l(P)))$. Since R/P is a prime affine PI algebra over F, $|R/P|=GK(R/P)$ by theorem 10.10 of Krause and Lenagan [6].

Observe that $l(P)$ is a finitely generated right module over R/P and that left multiplication by elements of R induces the injection $0\to R/l(l(P))\to \mathrm{End}_{R/P}(l(P))=T$. Thus

$$GK(R/l(l(P)))\leqq GK(T)\leqq GK(R/P),$$

with the second inequality following from proposition 2.9 of McConnell and Robson [8]. Finally we assemble the inequalities to obtain a contradiction:

$$|R|=|R(l(l(P))|\leqq GK(R/l(l(P)))\leqq GK(R/P)=|R/P|<|R|.$$

EXAMPLE. The ring

$$R=\left\{\begin{bmatrix} f(0) & g(x)\\ 0 & f(x)\end{bmatrix}\ \middle|\ f(x),\, g(x)\in k[x]\right\}$$

is right Noetherian, affine k-algebra which is not left Noetherian. R satisfies a polynomial identity and is irreducible and hence primary. R does not have an Artinian ring of quotients, but it is embeddable in a simple Artinian ring by the above criterion.

It is not known whether a right Noetherian, right fully bounded ring is embeddable in a right Artinian ring.

It is easy to see that if R is embeddable in an Artinian ring, then so are many of the relatives of R. For example if R embeds in the right Artinian ring S, then the $n \times n$ matrix ring $M_n(R)$ embeds in $M_n(S)$ and eRe embeds in eSe for any nonzero idempotent e. In particular any ring Morita equivalent to R embeds in a right Artinian ring. It can also be shown that a polynomial ring $R[x]$ over R also embeds in a right Artinian ring. In this spirit we offer the following theorem, and its corollary. We note that it is not necessary to assume that R is right Noetherian for these results to hold.

THEOREM 5. *Let R be a k-algebra which is embeddable in a simple Artinian ring, and let T be a k-algebra extension of R which is a finitely generated right R-module and which is an R-submodule of a free R-module. Then the algebra T can be embedded in a simple Artinian ring.*

PROOF. Since R can be embedded in a right Artinian k-algebra, R can be embedded in a simple Artinian ring S by theorem 7.13 of Schofield [9]. The homomorphism from R into S gives rise to a Sylvester module rank function λ, and since this homomorphism is an embedding we have that $\lambda(R/rR)=1$ implies that $r=0$. From the discussion on page 96 of Schofield we know that λ is defined by $\lambda(M)=d(M\otimes_R S)/d(S)$ where $d(X)$ denotes the length of a finitely generated right module over the simple Artinian ring S.

We observe that if I is a right ideal of R such that $\lambda(R/I)=1$, then $I=0$. To see this let $r\in I$, and consider the exact sequence

$$I/rR \to R/rR \to R/I \to 0.$$

Since λ is a Sylvester rank function we have that $1=\lambda(R/I)\leqq\lambda(R/rR)\leqq 1$ and so $\lambda(R/rR)=1$. Hence $r=0$ and so $I=0$.

We next extend λ to a Sylvester module rank function λ^* over the k-algebra T. If M is a finitely generated right T-module, then M is a finitely generated right R-module. We define

$$\lambda^*(M)=\frac{1}{\lambda(T)}\lambda(M).$$

It is clear that when λ^* is restricted to finitely presented right T-modules it is a

Sylvester module rank function over T. By Schofield's theorem λ^* gives rise to a homorphism from T into a simple Artinian ring. The kernel K of this homomorphism is $K = \{t \in T \mid \lambda^*(T/tT) = 1\}$. Our proof will be complete if we show that K is 0.

Now $\lambda^*(T/tT) = 1$ if and only if $\lambda(T/tT) = \lambda(T)$, which holds if and only if $d(T/tT \otimes_R S) = d(T \otimes_R S)$, and this equality holds precisely when the canonical image of $tT \otimes_R S$ is zero in $T \otimes_R S$. Since T is an R-submodule of a free R-module we have that $tT \subseteq T \subseteq R \oplus \cdots \oplus R = R^{(n)}$. We wish to show that $M = tT$ is 0. Suppose to the contrary that $M \subseteq R^{(n)}$, and that the canonical image of $M \otimes_R S$ is zero in $R^{(n)} \otimes_R S$, but $M \neq 0$. Say $0 \neq m = (r_1, \dots, r_n) \in M$. Without loss of generality we may assume $r_1 \neq 0$. Let I be the image of the projection of M onto the first component of $R^{(n)}$. Then $0 \neq r_1 \in I$ and I is right ideal of R. Since the canonical image of $M \otimes_R S$ is zero in $R^{(n)} \otimes_R S$, it follows that the canonical image of $I \otimes_R S$ is zero in $R \otimes_R S$ and so $d(R/I \otimes_R S) = d(R \otimes_R S)$. Hence $\lambda(R/I) = 1$ and therefore $I = 0$. With this contradiction the proof is complete.

We apply this theorem to the skew polynomial ring $R[x, \sigma]$ where σ is an automorphism of R of finite order and multiplication is given by $xr = \sigma(r)x$.

COROLLARY 6. *Let R be a k-algebra which is embeddable in a simple Artinian ring. Let σ be an automorphism of R of finite order. Then the skew polynomial ring $R[x, \sigma]$ is embeddable in a simple Artinian ring.*

PROOF. Suppose σ has order n, and that R embeds in the simple Artinian ring S.

The subring $R[x^n, \sigma]$ of $R[x, \sigma]$ is an ordinary polynomial ring and $R[x^n, \sigma] \subseteq S[x^n]$ which embeds in a simple Artinian ring. $R[x, \sigma]$ is a finitely generated free module over $R[x^n, \sigma]$ and thus $R[x, \sigma]$ is embeddable in a simple Artinian ring by Theorem 5.

If we remove the hypothesis that σ is of finite order we do not know whether the corollary still holds.

EXAMPLE. Let A_1 be the Weyl algebra $\mathbf{C}\{x, y \mid xy - yx = 1\}$ and $M = A_1/xA_1$. The ring

$$R = \begin{bmatrix} \mathbf{C} & M \\ 0 & A_1 \end{bmatrix}$$

of Small [11] is a right Noetherian ring which is not embeddable in a right or left

Artinian ring, but R is a finite module over $\mathbf{C}\oplus A_1$. It is worth noting that the Sylvester kernel

$$K=\begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}$$

is the nil radical and is not an annihilator ideal.

4. An Example

In Theorems 3 and 4 our embedding is achieved via a Sylvester rank function that is induced from the reduced rank function on the ring. In general a right Noetherian ring R can be written as a subdirect sum of a finite set of irreducible right Noetherian factor rings R_i. Thus if we can embed each R_i into a right Artinian ring, we will be able to embed R as well. If R were a k-algebra then it would admit a Sylvester module rank function. We will now provide an example of a subdirectly irreducible Noetherian k-algebra R which embeds in a simple Artinian k-algebra, and thus admits a Sylvester module rank function, but for which this Sylvester rank function is not induced by the reduced rank functions on R, or on any factor ring of R.

The ring essentially appears as example 3.1 of Blair [1]. Replace $A=\mathbf{R}[x]$ in that example by $\mathbf{R}[x]$ localized at the prime ideal (x^2+1). Then A is a commutative, local, principal ideal domain with maximal ideal P, for which there exists a monomorphism i: $A\to A/P$. Let ϕ: $A\to A/P$ be the natural map and set

$$R=\left\{\begin{bmatrix}a & m\\ 0 & a\end{bmatrix}\,\middle|\, a\in A,\ m\in A/P\right\}.$$

Define multiplication on R by

$$\begin{bmatrix}a & m\\ 0 & a\end{bmatrix}\begin{bmatrix}b & n\\ 0 & b\end{bmatrix}=\begin{bmatrix}ab & i(a)n+m\phi(b)\\ 0 & ab\end{bmatrix}.$$

The right ideals of R are $Q\supset Q^2\supset\cdots\supset\bigcap_{i=1}^{\infty}Q^i=N\supset 0$ where

$$Q=\left\{\begin{bmatrix}a & m\\ 0 & a\end{bmatrix}\,\middle|\, a\in P,\ m\in A/P\right\}\quad\text{and}\quad N=\left\{\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}\,\middle|\, m\in A/P\right\}.$$

R is right Noetherian, subdirectly irreducible of k-algebra where $k=\mathbf{Q}$ and R is embeddable in a simple Artinian k-algebra via f: $R\to M_2(A/P)$ where

$$f\left(\begin{bmatrix}a & m\\ 0 & a\end{bmatrix}\right)=\begin{bmatrix}i(a) & m\\ 0 & \phi(a)\end{bmatrix}.$$

Let λ be the Sylvester rank function associated to f. Then $\ker f = \{r \in R \mid \lambda(R/rR) = 1\} = 0$.

Let ρ be the reduced rank function of R. Then

$$\{r \in R \mid \rho(R/rR) = \rho(R)\} = \{r \in R \mid \text{for all } s \in R \text{ there exists } c \in C(N) \text{ such that } rsc = 0\} = N.$$

Let I be any nonzero ideal of R, and let ρ_I be the reduced rank function on R/I. Let $\bar{\rho}_I$ be the Sylvester rank function induced by ρ_I; that is, $\bar{\rho}_I(M) = \rho_I(M \otimes_R R/I)/\rho_I(R/I)$.

For any $r \in N$ we have $R/rR \otimes_R R/I = R/rR \otimes_{R/rR} R/I \simeq R/I$ and $R \otimes_R R/I \simeq R/I$ as R/I-modules. Thus $\bar{\rho}_I(R/rR) = 1$ and so $\bar{\rho}_I(R) = 1$, and $\bar{\rho}_I$ cannot distinguish between R/rR and R. It is also the case that ρ cannot distinguish between R/rR and R for $r \in N$. Since $\lambda(R/rR) \neq \lambda(R)$ for any $r \neq 0$ we have that λ cannot be a function of ρ and ρ_I. In particular λ is not a linear combination, sup or inf of reduced rank functions ρ and ρ_I.

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